$$\varphi = y^{3n-2} f_0(\zeta) + y^{5n-4} (\gamma+1)^{-1/3} f_1(\zeta) + y^{7n-6}(\gamma+1)^{-2/3} f_2(\zeta) + \ldots$$

Using (6), (8) and (10) we obtain the form f_1, f_2 (a prime denotes differentiation with respect to ζ)

$$f_1 = a_1 f_0 f_0' + a_2 \zeta f_0'^2 \quad (\omega = 0,\ \omega = 1)$$
$$a_1 = A\,(3n-2) + B, \quad a_2 = -nA$$

$$f_2 = b_1 \zeta^3 f_0'^2 + b_2 \zeta^2 f_0 f_0' + b_3 \zeta f_0^2 + b_4 \zeta f_0'^3 + b_5 f_0 f_0'^2 + b_6 \zeta f_0 f_0' f_0'' + b_7 f_0'^3 f_0'' + b_8 f_0^2 f_0'' \quad (\omega = 0)$$
$$b_1 = n^2 D/2 + (-10n+9) n^3 C/(2H)$$
$$b_2 = (-6n+4) b_1/n, \quad b_3 = (3n-2)^2 b_1/n^2$$
$$b_4 = 11n\,(n-1)A^2/2 - nAB + D/6 + (29n-24)nC/(6H)$$
$$b_5 = (5n-4)AB + 5\,(3n-2)(-n+1)A^2/2 + B^2 + (3n-2)(-3n+3)C/(2H)$$
$$b_6 = 2a_1 a_2, \ b_7 = 2A^2, \ b_8 = a_1^2/2$$
$$H = (7n-6)(4n-3)$$

The method of expanding the solution of (1) in a series in selfsimilar components is widely used, beginning with /8/, but the form of the corrections f_1, f_2 was found only for some particular values of n.

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ON MATHEMATICAL MODELS OF MAGNETIC FLUIDS*

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A system of equations describing models of magnetic fluids (MF) with internal angular momentum in a magnetic field is studied. Linearized equations and their solutions in the form of spin waves and magnetosonic waves are given. The high-frequency magnetic susceptibility tensor of the fluid is calculated and the frequencies of homogeneous magnetic resonance are determined. The connection between the spin and acoustic waves in MF is governed by the presence, in the internal energy of the fluid, of terms with vorticity vector and deformation rate tensor (determining, in particular, the hydromagnetic energy). Various existing models used to describe ferromagnetic fluids (FMF) are discussed. Relaxation models of MF are studied and used to obtain the solutions of problems of plane Couette flow and cylindrical Poiseuille flow. A new expression for the effective viscosity of the MF is obtained.

Several different models of MF are known. The simplest model /1/ describes paramagnetic fluids and certain types of the FMF in quasistationary magnetic fields quite well. However, in a number of important cases the above model cannot be used (e.g. at high frequencies of the magnetic field and for FMF at high volume concentrations of ferromagnetic particles with a

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fairly high energy of magnetic anisotropy). Because of this, other models of MF have received wide use in which the processes of relaxation of the magnetization and internal angular momenta were taken into account /2-9/.

1. Equations for the MF. The models of fluids taking internal angular momentum into account are introduced naturally on the basis of the Cosser continuum. We shall consider the following system of equations written in the variables of the universal observer coordinate system:

$$\rho \frac{d}{dt} \frac{1}{\rho} g^{\alpha}_{(m)} = \nabla_{\beta} P^{\alpha\beta}_{(m)} + \frac{1}{2} (M_{\lambda} \nabla^{\alpha} H^{\lambda} - H^{\lambda} \nabla^{\alpha} M_{\lambda}) + Q^{\alpha} \tag{1.1}$$

$$\rho \frac{d}{dt} \left(\frac{1}{g\rho} M^{\alpha} + I\Omega^{\alpha} \right) + \nabla_{\beta} A^{\alpha\beta} = \varepsilon^{\alpha\beta\lambda} M_{\beta} \left(B_{\lambda} - \rho \frac{\partial U_m}{\partial M^{\lambda}} - L_{\lambda} \right) + R^{\alpha}$$

$$B_{\alpha} - \rho \partial U_m / \partial M^{\alpha} + g^{-1} \Omega_{\alpha} = L_{\alpha}, \quad T = \partial U_m / \partial s$$

$$\operatorname{div} (H + 4\pi M) = 0, \ \operatorname{rot} H = 0, \ d\rho/dt + \rho \operatorname{div} v = 0$$

$$\rho T \frac{ds}{dt} = - \partial_{\alpha} q^{\alpha} + \tau^{\alpha\beta} e_{\alpha\beta} + L^{\alpha} \left(\rho \frac{d}{dt} \frac{1}{\rho} M_{\alpha} - [\omega, M]_{\alpha} \right) - R^{\alpha} (\Omega_{\alpha} - \omega_{\alpha}) - A^{\alpha\beta} \nabla_{\beta} \Omega_{\alpha}$$

(Maxwell's equations are written without taking into account the conduction and displacement currents, i.e. in the magnetostatic approximation). The components of the vector of volume density of the momentum of the fluid $g^{\alpha}_{(m)}$ and stress tensor $p^{\alpha\beta}_{(m)}$ in (1.1) are given by the relation

$$g^{\alpha}_{(m)} = \rho v^{\alpha} + \nabla_{\beta} \frac{\partial \rho U_m}{\partial \nabla_{\beta} v_{\alpha}} \tag{1.2}$$

$$P^{\alpha\beta}_{(m)} = - p g^{\alpha\beta} - \rho \frac{\partial U_m}{\partial \nabla_{\beta} v_{\lambda}} \nabla^{\alpha} v_{\lambda} + \tau^{\alpha\beta} \mp \frac{1}{2} (-\varepsilon^{\alpha\beta\lambda} R_{\lambda} + M^{\alpha} L^{\beta} - M^{\beta} L^{\alpha})$$

$$p = \rho^2 \frac{\partial U_m}{\partial \rho} + \rho M^{\alpha} \frac{\partial U_m}{\partial M^{\alpha}} - \frac{1}{2} M^{\alpha} B_{\alpha}$$

In (1.1) and (1.2) Q^{α} are the components of the vector of external mass forces acting on the fluid (e.g. the force of gravity), B_{α} are the components of the magnetic induction vector, v^{α} are the components of the velocity vector of the individual points of the fluid, ρ is the mass density of the fluid, d/dt denotes the total derivative with respect to time t, ∇_{β} denotes the covariant derivative calculated relative to the universal observer coordinate system with variables x^{α} ($\alpha = 1, 2, 3$) and metric tensor determined by the covariant components $g^{\alpha\beta}$, Ω_{α} are the components of the internal rotation vector (the angular velocity vector of the orthonormal bases of the Cosser continuum), ω_{α} are the components of the vorticity vector, $e_{\alpha\beta}$ are the components of the deformation rate tensor, s is the specific density of the entropy, T is temperature, g, I are given constant coefficients, $\varepsilon^{\alpha\beta\lambda}$ are the components of the Levi-Civita pseudotensor, and U_m is a given differentiable function of the parameters ρ, s, M_{α}, $\nabla_{\beta} v_{\alpha}$ (representing part of the internal energy of the fluid). In the case when the function U_m does not depend on the velocity gradients, the equations discussed here were obtained in /3/ with help of the variational equation. The Lagrangian Λ corresponding to Eqs.(1.1), (1.2) has the form

$$\Lambda = \frac{1}{2} \rho v^2 - \frac{1}{8\pi} B^2 + B_{\alpha} M^{\alpha} + \frac{1}{g} M^{\alpha} \Omega_{\alpha} + \frac{1}{2} \rho I \Omega^2 - \rho U_m (\rho, s, M_{\alpha}, \nabla_{\beta} v_{\alpha})$$

The relaxation term R^{α} of the equations of angular momentum in (1.1), the relaxation term L^{α} of the magnetization equation, the components of the viscous stress tensor $\tau^{\alpha\beta}$, the components of instantaneous stress tensor $A^{\alpha\beta}$ and the components of the heat flux vector q^{α} governing the dissipative processes in the fluid, can be determined (according to Onsager) by the relation

$$\tau^{\alpha\beta} = \tau^{\alpha\beta\lambda\theta} e_{\lambda\theta}, \quad q^{\alpha} = - \varkappa^{\alpha\beta} \nabla_{\beta} T, \quad A^{\alpha\beta} = - a^{\alpha\beta\lambda\theta} \nabla_{\lambda} \Omega_{\theta} \tag{1.3}$$

$$L^{\alpha} = \theta^{\alpha\beta} \left(\rho \frac{d}{dt} \frac{1}{\rho} M_{\beta} - [\omega, M]_{\beta} \right), \quad R^{\alpha} = - r^{\alpha\beta} (\Omega_{\beta} - \omega_{\beta})$$

in which the coefficients $\tau^{\alpha\beta\lambda\theta}$, $a^{\alpha\beta\lambda\theta}$, $\varkappa^{\alpha\beta}$, $r^{\alpha\beta}$, $\theta^{\alpha\beta}$ are chosen so as to ensure that the internal generation of entropy is not negative; all these coefficients can be functions of the defining parameters of the fluid and field. In particular, in the isotropic case we can assume that the quantities $\theta^{\alpha\beta}$, $r^{\alpha\beta}$ are

$$\theta^{\alpha\beta} = \theta (g^{\alpha\beta} - n^{\alpha} n^{\beta}) + \theta_{\parallel} n^{\alpha} n^{\beta} + \theta_{*} \varepsilon^{\alpha\beta\lambda} n_{\lambda} \tag{1.4}$$

$$r^{\alpha\beta} = \frac{1}{g^2} \left[\frac{1}{\tau} (g^{\alpha\beta} - n^{\alpha} n^{\beta}) + \frac{1}{\tau_{\parallel}} n^{\alpha} n^{\beta} + \frac{1}{\tau_{*}} \varepsilon^{\alpha\beta\lambda} n_{\lambda} \right]$$

where $\theta, \theta_\parallel, \theta_*, \tau, \tau_\parallel, \tau_*$ are scalar coefficients with a dimensions of time (relaxation times) and $n^\alpha = M^\alpha / |M|$ are the components of the unit vector directed along the fluid magnetization vector.

If $1/g \neq 0$ and the coefficients $\theta^{\alpha\beta}$ are independent of Ω, then the third equation of (1.1) will enable us to eliminate the internal rotation vector from the remaining equations. In particular, after eliminating Ω the equation of angular momentum takes the form

$$\rho \frac{d}{dt}\left(\frac{1}{\rho} M^\alpha - Ig^2 H^{*\alpha}\right) + \nabla_\beta \left(g^2 a^{\alpha\beta\lambda\theta} \nabla_\lambda H_\theta^*\right) = g\,[M, H^*]^\alpha + g^2 r^{\alpha\beta} (H_\beta^* + g^{-1}\omega_\beta) \tag{1.5}$$

where $H^{*\alpha}$ are the components of the effective magnetic field vector

$$H^{*\alpha} = B^\alpha - \rho \frac{\partial U_m}{\partial M_\alpha} - \theta^{\alpha\beta}\left(\rho \frac{d}{dt} \frac{1}{\rho} M_\beta - [\omega, M]_\beta\right)$$

In some theories expressions (1.3) for R^α, L^α are replaced by the following Onsager relations:

$$L^\alpha = \theta^{\alpha\beta}\left(\rho \frac{d}{dt} \frac{1}{\rho} M_\beta - [\Omega, M]_\beta\right) \tag{1.6}$$
$$R^\alpha - \varepsilon^{\alpha\beta\lambda} M_\beta L_\lambda = - r^{\alpha\beta} (\Omega_\beta - \omega_\beta)$$

If the quantities $\partial U_m / \partial M_\alpha$ are proportional to M^α then the quantities L^α can also be found using the expression (without changing the internal generation of the entropy)

$$L^\alpha = \theta^{\alpha\beta}\left(\rho \frac{d}{dt} \frac{1}{\rho} M_\beta - g\,[M, H]_\beta\right) \tag{1.7}$$

The basic difference between the equations discussed here and the corresponding equations for isotropic fluids /9, 10/ is connected with the fact that thermodynamic fluxes R^α, L^α are expressed by (1.3) instead of (1.6), (1.7) and the expressions $dM/dt - [\Omega, M]$ in /10/ or $dM/dt - g\,[M, H]$ in /9/ for the thermodynamic forces are replaced by the expressions $dM/dt - [\omega, M]$.

Eqs.(1.1)-(1.3) describe the models of viscous compressible and magnetizable fluids possessing the internal angular momentum

$$K^\alpha = g^{-1} M^\alpha + \rho I \Omega^\alpha$$

In the real FMF the internal angular momentum is determined by the ferromagnetic single-domain particles dispersed in it, which have a natural, spin-generated angular momentum K_s (related to the magnetic moment of the particles), and the moment K_Ω governed by the mechanical rotation of these particles. As was shown in /11/, the angular momentum K_Ω determined by the rotation of the domains is several orders of magnitude less than the spin moment of the domains in FMF. ($K_\Omega = 10^{-8} K_s$ for domains of diameter $\sim 10^{-6}$ cm., even at large angular velocities $\Omega_0 \sim 10^2$ sec^{-1} of the domains.

Eqs.(1.1)-(1.3) considered here can be used to describe the FMF in high-frequency magnetic fields, and they sharpen the Neuringer-Rosenzweig equations /1/ in the quasistatic field by taking into account magnetic relaxation processes in the FMF.

In the general case, the internal rotation vector Ω in (1.1)-(1.3) cannot be connected with any real relations in the medium (including the FMF) when we have the spin momentum $g^{-1}M$. Indeed, e.g. when $I = 0$, Eqs.(1.1)-(1.3) can be used to describe paramagnetic or diamagnetic fluids. Here it is obvious that in such fluids the vector Ω is different from zero, but does not describe any real rotations in these media.

If we use the theory of FMF to determine the vector Ω, with the spin resulting from averaging the angular velocity vectors of the particles in FMF, as is done in /9, 10/, then passage to the limit in the equations of /10/ for the isotropic FMF corresponding to freezing the particles into the fluid (as $\tau_s \to 0$ in Eq.(2.33) or (2.36) in the notation of /10/) yields, for the spin angular velocity vector $g^{-1}M$, not the precession equation (the Landau-Lifshitz equation) which should be the case, but a special relaxation equation. This shows that the equations for the isotropic FMF with a spin given in /10/ and discussed here, based on the concept of Ω shown above, are incorrect from the physical point of view.

2. Linearized equations.

Let us investigate the isentropic motions of a fluid described by Eqs.(1.1)-(1.3), corresponding to the function U_m of the form

$$\rho U_m = \rho U_m^0 (\rho, M) + 2\pi M^2 + \gamma M^\alpha \omega_\alpha + {}^1/_2\, \nu_1 M^\alpha M^\beta e_{\alpha\beta} + {}^1/_2 \nu_2 M^2 e_\alpha{}^\alpha \tag{2.1}$$

where $M = (M_\alpha M^\alpha)^{1/2}$ is the modulus of the magnetizability vector and ν_1, ν_2, γ are given constants. The coefficients $\theta^{\alpha\beta}, r^{\alpha\beta}$ in (1.3) are found from Eqs.(1.4) in which $\tau_*^{-1} = \theta_* = 0$, and we adopt the following expressions for $\tau^{\alpha\beta\lambda\theta}, a^{\alpha\beta\lambda\theta}$:

$$\tau^{\alpha\beta\lambda\theta} = \mu\,(g^{\alpha\lambda} g^{\beta\theta} + g^{\alpha\theta} g^{\beta\lambda}) + \lambda g^{\alpha\beta} g^{\lambda\theta} \tag{2.2}$$
$$a^{\alpha\beta\lambda\theta} = D g^{-2} g^{\alpha\beta} g^{\lambda\theta}$$

where λ, μ are the coefficients of viscosity and D is a constant. In the case in question the components of the momentum density vector $g^{\alpha}_{(m)}$, of the stress tensor $P^{\alpha\beta}_{(m)}$ and of the effective magnetic field vector $H^{*\alpha}$ in Eqs. (1.1), (1.6), are determined as follows:

$$
\begin{aligned}
&H^{*\alpha} = H^{\alpha} - \chi^{-1}M^{\alpha} - \gamma\omega^{\alpha} - \nu_1 e^{\alpha\beta}M_{\beta} - \nu_2 M^{\alpha}e_{\lambda}{}^{\lambda} - \\
&\quad \theta^{\alpha\beta}(dM_{\beta}/dt - [\omega, M]_{\beta} + M_{\beta}e_{\lambda}{}^{\lambda}) \\
&g^{\alpha}_{(m)} = \rho v^{\alpha} - {}^1/_2\,\gamma\,\mathrm{rot}^{\alpha}M + {}^1/_2\,\nabla_{\lambda}(\nu_1 M^{\alpha}M^{\lambda} + \nu_2 M^2 g^{\alpha\lambda}) \\
&P^{\alpha\beta}_{(m)} = -pg^{\alpha\beta} + {}^1/_2\,\gamma\varepsilon^{\beta\lambda\mu}M_{\lambda}\nabla^{\alpha}v_{\mu} - {}^1/_2\,\nu_1 M^{\beta}M^{\lambda}\nabla^{\alpha}v_{\lambda} - {}^1/_2\,\nu_2 M^2\nabla^{\alpha}v^{\beta} + \\
&\quad \mu(\nabla^{\alpha}v^{\beta} + \nabla^{\beta}v^{\alpha}) + \lambda g^{\alpha\beta}\nabla_{\lambda}v^{\lambda} + {}^1/_2(-\varepsilon^{\alpha\beta\lambda}R_{\lambda} + M^{\alpha}L^{\beta} - M^{\beta}L^{\alpha})
\end{aligned}
\tag{2.3}
$$

For the magnetic susceptibility χ and pressure p in the definitions (2.3) we have

$$
\chi = \left(\frac{\rho}{M}\frac{\partial U_m{}^{\circ}}{\partial M}\right)^{-1}, \quad p = \rho^2\frac{\partial U_m{}^{\circ}}{\partial\rho} + \rho M\frac{\partial U_m{}^{\circ}}{\partial M} + {}^1/_2\,\nu_1 M^{\alpha}M^{\beta}e_{\alpha\beta} + {}^1/_2\,\nu_2 M^2 e_{\lambda}{}^{\lambda} - {}^1/_2\,M^{\alpha}H_{\alpha}
\tag{2.4}
$$

We will further assume that the observer coordinate system with variables x^{α} is Cartesian. Let $M_0, H_0, \rho_0, v_0 = 0$ be constant values of the parameters of the fluid in the state of equilibrium. Assuming that the functions $M(x^{\alpha}, t)$, $H(x^{\alpha}, t)$, $\rho(x^{\alpha}, t)$, $v(x^{\alpha}, t)$ vary very little relative to their equilibrium values,

$$
M = M_0 + \mu, \quad H = H_0 + h, \quad \rho = \rho_0 + \rho_1
$$

we obtain the following expression for the function U_m° up to second-order infinitesimals:

$$
\rho U_m{}^{\circ} = {}^1/_2\,\rho_0^{-1}a_*{}^2\rho_1^2 + {}^1/_2\,\beta^{\alpha\beta}\mu_{\alpha}\mu_{\beta} + bn^{\alpha}\mu_{\alpha}\rho_1 + \beta_1 n^{\alpha}\mu_{\alpha} + A\rho_1 + \mathrm{const}
\tag{2.5}
$$

Here

$$
\begin{aligned}
&\beta^{\alpha\beta} = \beta_1\delta^{\alpha\beta} + \beta_2 n^{\alpha}n^{\beta}, \quad b = \left(\frac{\partial U_m{}^{\circ}}{\partial M} + \rho\frac{\partial^2 U_m{}^{\circ}}{\partial\rho\partial M}\right)_0 \\
&\beta_1 = \left(\frac{\rho}{M}\frac{\partial U_m{}^{\circ}}{\partial M}\right)_0, \quad \beta_2 = \left(\rho\frac{\partial^2 U_m{}^{\circ}}{\partial M^2} - \frac{\rho}{M}\frac{\partial U_m{}^{\circ}}{\partial M}\right)_0 \\
&n^{\alpha} = \frac{M_0{}^{\alpha}}{M_0}, \quad a_*{}^2 = \left(\rho\frac{\partial^2\rho U_m}{\partial\rho^2}\right)_0, \quad A = \left(U_m + \rho\frac{\partial U_m}{\partial\rho}\right)_0
\end{aligned}
$$

The symbol $(\ )_0$ means that the expressions in parenthesis are taken at the equilibrium values of the parameters.

Taking into account relations (2.1)-(2.5), we linearize equations (1.1), (1.6) from which the function $\Omega(x^{\alpha}, t)$ has been eliminated with help of the equation for magnetizability in (1.1)

$$
\begin{aligned}
&\partial(\mu^{\alpha} - \rho_0 I g^2 h^{*\alpha})/\partial t + D\Delta h^{*\alpha} + M_0{}^{\alpha}\partial_{\lambda}v^{\lambda} = g[M_0, h^*]^{\alpha} + g^2 r^{\alpha\beta}(h_{\beta}{}^* + g^{-1}\omega_{\beta}) \\
&(\partial/\partial t)\{\rho_0 v^{\alpha} - {}^1/_2\,\gamma\,\mathrm{rot}^{\alpha}\mu + \partial_{\lambda}[{}^1/_2\nu_1(M_0{}^{\lambda}\mu^{\alpha} + M_0{}^{\alpha}\mu^{\lambda}) + \nu_2 M_0{}^{\beta}\mu_{\beta}\delta^{\alpha\lambda}]\} = \\
&\quad -\partial^{\alpha}p_* + \mu\Delta v^{\alpha} + (\lambda + \mu)\partial^{\alpha}\partial_{\beta}v^{\beta} + M_0{}^{\lambda}\partial^{\alpha}h_{\lambda} + {}^1/_2\partial_{\beta}[\theta M_0{}^{\alpha}(\partial\mu^{\beta}/\partial t - \\
&[\omega, M_0]^{\beta}) - \theta M_0{}^{\beta}(\partial\mu^{\alpha}/\partial t - [\omega, M_0]^{\alpha}) - g^{-1}\varepsilon^{\alpha\beta\lambda}r_{\lambda\theta}(h^{*\theta} + g^{-1}\omega^{\theta}) \\
&\mathrm{rot}\,h = 0, \ \mathrm{div}(h + 4\pi\mu) = 0, \ \partial\rho_1/\partial t + \rho_0\,\mathrm{div}\,v = 0
\end{aligned}
\tag{2.6}
$$

Here

$$
\begin{aligned}
&h^{*\alpha} = h^{\alpha} - \beta^{\alpha\beta}\mu_{\beta} - \gamma\omega^{\alpha} - \nu_1 M_{\lambda}{}^{\circ}e^{\alpha\lambda} - \nu_2 M_0{}^{\alpha}\partial_{\lambda}v^{\lambda} - bn^{\alpha}\rho_1 - \\
&\quad [\theta(\delta^{\alpha\lambda} - n^{\alpha}n^{\lambda}) + \theta_{\parallel}n^{\alpha}n^{\lambda}](\partial\mu_{\lambda}/\partial t - [\omega, M_0]_{\lambda} + M_{\lambda}{}^{\circ}\partial_{\beta}v^{\beta}) \\
&p_* = (a_*{}^2 + bM_0)\rho_1 + [\rho_0 b + M_0(\beta_1 + \beta_2)]n^{\alpha}\mu_{\alpha} + \gamma M_0{}^{\alpha}\omega_{\alpha} + \\
&\quad \nu_1 M_0{}^{\alpha}M_0{}^{\beta}e_{\alpha\beta} + \nu_2 M_0{}^2\partial_{\alpha}v^{\alpha}
\end{aligned}
$$

3. Spin waves. Let us consider the spin waves in MF in the linearized formulation, i.e. the small adiabatic oscillations in the magnetization of the fluid, neglecting the acoustic waves. In accordance with the formulation of the problem we shall write the initial equations in the form

$$
\begin{aligned}
&\mathrm{rot}\,h = 0, \quad \mathrm{div}(h + 4\pi\mu) = 0 \\
&\partial(\mu^{\alpha} - \rho_0 I g^2 h^{*\alpha})/\partial t + D\Delta h^{*\alpha} = g[M^0, h^*]^{\alpha} + [\tau^{-1}(\delta^{\alpha\beta} - n^{\alpha}n^{\beta}) + \\
&\quad \tau_{\parallel}^{-1}n^{\alpha}n^{\beta}]h_{\beta}{}^* \\
&h^{*\alpha} = h^{\alpha} - \beta^{\alpha\beta}\mu_{\beta} - [\theta(\delta^{\alpha\lambda} - n^{\alpha}n^{\lambda}) + \theta_{\parallel}n^{\alpha}n^{\lambda}]\,\partial\mu_{\lambda}/\partial t
\end{aligned}
\tag{3.1}
$$

Assuming that

$$
\mu = \mu_* \exp[i(-\omega t + k_{\alpha}x^{\alpha})], \quad h = h_* \exp[i(-\omega t + k_{\alpha}x^{\alpha})]
$$

where ω is the frequency of the wave and k_{α} are components of the wave vector, we obtain from (3.1)

$$\mu^{\alpha} = \chi^{\alpha\beta} h_{\beta}, \quad h^{\alpha} = -4\pi k^{-2} k^{\alpha} k_{\beta} \mu^{\beta} \tag{3.2}$$

We define the components of the high-frequency magnetic susceptibility tensor $\chi^{\alpha\beta}$ by the relation

$$\chi^{\alpha\beta} = \chi_1 \delta^{\alpha\beta} + \chi_2 n^{\alpha} n^{\beta} + i\chi_3 n_{\lambda} \varepsilon^{\alpha\beta\lambda} \tag{3.3}$$

$$\begin{gathered} \chi_1 = X_- + X_+, \; \chi_3 = X_- - X_+ \\ X_{\pm} = {}^1/_2 \, [\beta_1 - i\omega\theta + \omega \, (i/\tau - iDk^2 \pm gM_0 + \omega\rho_0 I g^2)^{-1}]^{-1} \\ \chi_2 = -\chi_1 + [\beta_1 + \beta_2 - i\omega\theta_{\parallel} + \omega \, (i/\tau_{\parallel} - iDk^2 + \omega\rho_0 I g^2)^{-1}]^{-1} \end{gathered} \tag{3.4}$$

The following dispersion equation for the spin waves follows from Eqs.(3.1):

$$k^2 + 4\pi\chi^{\alpha\beta} k_{\alpha} k_{\beta} = 0, \; \text{or} \; k^2 + 4\pi \, [\chi_1 k^2 + \chi_2 \, (k_{\alpha} n^{\alpha})^2] = 0 \tag{3.5}$$

In the non-dissipative approximation (when $\theta = \theta_{\parallel} = D = \tau_{\parallel}^{-1} = \tau^{-1} = 0$) the dispersion Eq.(3.5) yields the following expression for the frequency of the spin wave (which depends only on the direction of the wave vector and is independent of its modulus):

$$\omega = \pm \frac{gM_0}{1 + \eta\beta_1} \left[\beta_1^2 + \frac{4\pi\beta_1 \sin^2\psi}{1 + \eta\,(4\pi + \beta_1) - 4\pi\eta^2 \cos^2\psi \, [1 + \eta\,(\beta_1 + \beta_2)]^{-1}\beta_2}\right]^{1/2} \tag{3.6}$$

Here ψ is the angle between the wave vector and the vector M_0, $\eta = \rho_0 I g^2$.

Let us now consider a finite volume of the MF bounded by an ellipsoidal surface in an external (lateral) variable magnetic field varying with time according to the law $e^{-i\omega t}$.

We shall assume that the wavelength of the lateral field is much greater than the characteristic dimension l of the fluid ellipsoid, so that the field can be assumed to be homogeneous at distances of the order of l. In this case a homogeneous magnetic field is generated within the fluid and the magnetic resonance frequencies are given by the equation

$$\det\,(1 + 4\pi N\chi) = 0 \tag{3.7}$$

where 1 is a three-dimensional unit matrix, N is the matrix of the components of the tensor of demagnetizing coefficients, and χ is the matrix of components of the high-frequency magnetic susceptibility tensor calculated without dissipation: $\theta = \theta_{\parallel} = D = \tau^{-1} = \tau_{\parallel}^{-1} = 0$.

Assuming that the principal axes of the fluid ellipsoid are directed along the axes of the coordinate system ($N = \mathrm{diag}\,(N_1, N_2, N_3)$), we obtain from (3.7) the following expression for the resonance frequency:

$$\omega_* = gM_0\,(\xi_1\xi_2)^{1/2}, \quad \xi_k = \frac{\beta_1 + 4\pi N_k}{1 + \rho_0 I g^2\,(\beta_1 + 4\pi N_k)} \tag{3.8}$$

If $N_1 = N_2 = N$ (e.g. for a sphere $N = 1/3$, and for a cylinder $N = 1/2$), then $\xi_1 = \xi_2 = \xi$ and the formula for ω_* becomes

$$\omega_* = gM_0\xi \tag{3.9}$$

We see from expression (3.8) that the frequency of uniform magnetic resonance depends on the coefficient I determining the energy of internal rotation. Estimating the quantity ω_* for the FMF we find, that taking into account the internal rotation energy can appreciably reduce the resonance frequency ω_* as compared with its magnitude in the corresponding magnetic solids. This is in qualitative agreement with the experimental data given in /12/.

4. Magnetoacoustic waves. We shall consider Eqs.(2.7), (2.8) in the non-dissipative approximation, i.e. for $\theta^{\alpha\beta} = r^{\alpha\beta} = D = \lambda = \mu = 0$. We shall seek a solution of these equations in the form of a harmonic wave propagating along the vector of constant magnetization of the fluid $k^{\alpha} = kn^{\alpha}$. In this case Eqs.(2.7) yield

$$\begin{gathered} \mu^{\alpha} = \chi^{\alpha\beta} h_{\beta} + \zeta^{\alpha\beta} v_{\beta}, \quad h^{\alpha} = -4\pi n^{\alpha} n_{\lambda} \mu^{\lambda}, \quad \rho_1 = \rho_0 \omega^{-1} k_{\alpha} v^{\alpha} \\ \{-i\omega\rho_0\delta^{\alpha\beta} + k^2 n^{\alpha} n^{\beta} \, [i\rho_0\,(a_0^2 + M_0 b)\,\omega^{-1} - M_0^2\,(\nu_1 + \nu_2)]\} \, v_{\beta} + \\ \{{}^1/_2 M_0 \nu_1 k\omega\delta^{\alpha\beta} + n^{\alpha} n^{\beta} k M_0 \cdot [({}^1/_2\,\nu_1 + \nu_2)\,\omega + i\,(\rho_0 b M_0^{-1} + 4\pi + \beta_1 + \beta_2)] + \\ {}^1/_2\,k\gamma\varepsilon^{\alpha\beta\lambda} n_{\lambda}\} \, \mu_{\beta} = 0 \end{gathered} \tag{4.1}$$

where the components $\chi^{\alpha\beta}$ are given by Eq.(3.3) in which we must put $\theta = \theta_{\parallel} = D = \tau^{-1} = \tau_{\parallel}^{-1} = 0$, and for $\zeta^{\alpha\beta}$ we have

$$\begin{gathered} \zeta^{\alpha\beta} = \zeta_1\delta^{\alpha\beta} + \zeta_2 n^{\alpha} n^{\beta} + i\zeta_3 \varepsilon^{\alpha\beta\lambda} n_{\lambda} \\ \zeta_1 = {}^1/_2 k\,(-iM_0\nu_1\chi_1 + \gamma\chi_3), \; \zeta_3 = {}^1/_2 k\,(-iM_0\nu_1\chi_3 + \gamma\chi_1) \\ \zeta_2 = -\zeta_1 - k\,(\chi_1 + \chi_2)[iM_0\,(\nu_1 + \nu_2) + \omega^{-1}\,(\rho_0 b - M_0/(\rho_0 I g^2))] \end{gathered}$$

The dispersion equations for transverse waves follow from (4.1)

$$\omega = \mp M_0 g \frac{\rho_0\beta_1 + {}^1/_4\,(\gamma^2 + \nu_1 M_0^2)\,k^2}{\rho_0 + \rho_0 I g^2\,[\rho_0\beta_1 + {}^1/_4\,(\gamma^2 + \nu_1 M_0^2)\,k^2]} \tag{4.2}$$

Putting $n^{\alpha} = (0, 0, 1))$ we obtain the following expressions for the transverse waves described by the dispersion Eq.(4.2):

$$\rho_1 = 0,\ \mu^{\alpha} = (\mu^1, \mu^2, 0),\ v^{\alpha} = (v^1, v^2, 0),\ h^{\alpha} = 0$$

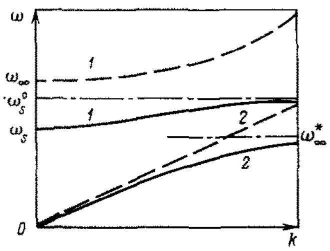
and the quantities μ, v are connected by the relations

$$v^1 \pm iv^2 = 0,\ \mu^1 \pm i\mu^2 = 0$$
$$v^1 \mp iv^2 = {}^1/_2 ik\rho_0^{-1} (\nu_1 M_0 \pm i\gamma)(\mu^1 \mp i\mu^2) \tag{4.3}$$

The signs in expressions (4.2) and (4.3) are matched. If $\nu_1 = \gamma = 0$, then the dispersion Eq.(4.2) will determine the frequency of the spin wave

$$\omega_s = \mp M_0 g\beta_1 (1 + \rho_0 I g^2 \beta_1)^{-1} \tag{4.4}$$

for which $v^1 = v^2 = v^3 = 0$. Thus the propagation of transverse acoustic waves through the fluid is due to the presence in the energy U_m of crossover terms with $M_{\alpha}, \omega_{\alpha}, e_{\alpha\beta}$, and, in partiuclar, the gyromagnetic energy $\gamma M^{\alpha}\omega_{\alpha}$.



The solid line *1* in the figure shows schematically the dispersion curve of the transverse waves. The dashed line *1* shows the dispersion curve for $I = 0$, with $\omega_s^{\circ} = - M_0 g\beta_1$, $\omega_{\infty} = -M_0/(\rho_0 I g)$.

Eqs.(4.1) also yield a dispersion equation for a longitudinal wave

$$\omega^2 = a^2k^2 (1 + e^2k^2)^{-1},\ e^2 = Ig^2M_0^2 (\nu_1 + \nu_2)^2 \lambda \tag{4.5}$$
$$a^2 = a_0^2 + M_0 b + \rho_0^{-1} (M_0 - \rho_0^2 I g^2 b)[\rho_0 b + M_0 (4\pi + \beta_1 + \beta_2)] \lambda$$
$$\lambda = [1 + \rho_0 I g^2 (4\pi + \beta_1 + \beta_2)]^{-1}$$

If $n^{\alpha} = (0, 0, 1)$, then the following relations hold for the longitudinal wave:

$$v^{\alpha} = (0, 0, v),\ \mu^{\alpha} = (0, 0, \mu),\ h^{\alpha} = (0, 0, -4\pi\mu)$$
$$\mu = k\omega^{-1} v\lambda \{M_0 - \rho_0 I g^2 [\rho_0 b + iM_0 (\nu_1 + \nu_2)\ \omega]\}$$

The dispersion curve of the longitudinal wave is shown in the figure with the solid line *2* (the dashed line *2* corresponds to $I = 0$, $\omega_{\infty}{}^* = a/e$ and ω^* is placed on the graph arbitrarily).

We see from Eq.(4.5) that the velocity of the longitudinal wave depends, in general, on the frequency of the wave (i.e. there is a dispersion of the velocity of sound).

5. On the effect of an increase in the viscosity of MF in a magnetic field. Below we discuss relaxation models of MF within whose framework the solution of the Couette and Poiseuille problems are given. The problems of plane Couette and Poiseuille flows were studied in /11/ within the framework of the model of MF with internal angular momentum related to the magnetization. The problem of plane Couette flow was studied within the framework of the theory of FMF with internal rotation, in /8/. As was noted in /9, 10/, the incremental increase in the coefficient of viscosity of the fluid in magnetic field in these problems, described using the model of MF /11/ and applied to the FMF, differs from the values obtained in the well-known theory of FMF with internal rotation /7/ by several orders of magnitude. In this connection we note at once that the equations in /11, 7/ describe different FMF, and therefore we cannot compare the incremental increase in viscosity given in these papers, as was done in /9, 10/.

It must be remembered that various types of FMF exist for which the effect of change in the viscosity in magnetic field is essentially different. For the FMF in which the disperse ferromagnetic particles have low energy of magnetic anisotropy, in which case the orientation of the particle and the direction of its magnetic moment are practically unrelated (super paramagnetism), the effect will be vanishingly small (the change in the value of the coefficient of viscosity will be a fraction of a percent of its value /13/). As we know, such FMF in quasistationary fields are well-described by the Neuringer-Rosenzweig (NR) equations /1/ which do not take into account the change in viscosity in the magnetic field at all. The model of the FMF in /11/ is a generalization of the NR model, and takes into account the spin-related internal angular momentum which really exists in FMF. Therefore, the equations of /11/ describe the FMF in a quasistationary field, in any case not less accurately than the NR equations, and can be used to describe the FMF in high-frequency magnetic fields when the NR equations become inapplicable. In the FMF in which the dispersed ferromagnetic particles are of sufficiently high energy of magnetic anisotropy and the magnetic moments are "frozen" into these particles, the effect of a change in viscosity in the magnetic field is more significant, and in some cases it must be taken into account. This is what happens in the equations used in /8/. However, when using the equations in /8/, we find that the maximum increment in the coefficient of viscosity in the Couette problem is small even when the parameters are optimal (in the limit magnetic field when the magnetic field and the vorticity vectors are orthogonal and the volume concentration of the particles in the fluid is high

$\varphi \sim 0{,}2$) and reaches $\sim$20–25% of the value of the coefficient of viscosity. In many cases (though not always) such FMF can be described by the equations of /1/ with sufficient accuracy. Thus the crux of the matter is that the incremental increase in the value of the coefficient of viscosity need only be taken into account for certain types of the FMF in fairly strong magnetic fields.

In this connection, the use in /8/ of the vector of internal rotation $I\Omega$ is not essential for describing the effect of an increase in the viscosity of FMF in a magnetic field, and the dependence of the increment in the value of the coefficient of viscosity on the coefficient I in /8/ is only related to the notation of the coefficients in the other (relaxation) terms of the momentum equation /5/.

Moreover, as we know /14/, the term containing the angular momentum $I\Omega$ in the momentum equation in the theory of FMF with internal rotation /7, 8/ can be comparable to the remaining terms of the momentum equation only at very high frequencies of the magnetic field, when the equations become inapplicable from the physical point of view (even if because of the need to take into account the appreciably larger /11/ spin moment of the domains in the FMF at much lower frequencies). Therefore, the use of the moment $I\Omega$ in existing theories /7, 8/ is not justified, although these theories have achieved a certain limited application.

The problem of taking into account the internal (gyromagnetic) angular momentum of the FMF was discussed in /10/. The basic assertions made by the authors in /10/, concerning the internal moments are that the equations of /11/, which take into account only the spin angular momentum, cannot be used to describe the FMF, since the terms containing the spin moment are small compared with other terms of the equations, and the attempt to use these equations to describe the hydrodynamic motion, as was asserted in /10/, leads to a disagreement with experimental data of FMF viscosity measurements. Both these assertions are false.

Indeed, when the terms containing the gyromagnetic ratio g are neglected, the equations with internal moment $g^{-1}M$ (and without taking into account the relaxation of the magnetization M) discussed in /10/, become in NR equations /1/ which, as we know and as the authors note in /10/, can be, and in fact are used in practice to describe the FMF. Thus the proof given by the authors in /10/ implies the opposite, namely that the equations of /4/ can be used to describe the FMF. As regards the assertions made by the authors in /10/, that the results of /11/ contradict experimental data on the viscosity of FMF, we should note that the experiments to which they refer deal with the FMF in which the particles have magnetic moments "frozen in" and therefore bear no relation to the FMF with "super paramagnetism" described by the equations of /11/.

It should also be noted that the proof that terms with internal spin moment in the equations of /11/ are small, given in /10/, was based on using, in the estimates given, the characteristic time $t \sim 1$ sec. and internal rotation $\Omega \sim 1\ \text{sec}^{-1}$. It is, however, clear that the characteristic time in the wave processes in FMF (described by the hydrodynamic equations!) can be 10^{-1}, 10^{-3} and 10^{-7} sec. and even shorter, a fact ignored in /10/, where the arguments are based on the condition that $t \sim 1$ sec. only. The statement in /10/ that they used in their estimates the same values of the parameters as in /11/, is incorrect: in /11/ a value of 100 sec^{-1} was adopted for the angular velocity and the domain in FMF, and no estimate was used at all for the time t.

It is precisely by taking into account the terms with internal moment $g^{-1}M$ (which are indeed small for the values of the parameters used in /10/, but become appreciable in high-frequency hydrodynamic processes), that the equations discussed in /11/ describe, for example, the well-known spin waves.

A theory taking into account the internal angular momenta in the mechanics of FMF (including those connected with iternal rotation), was developed in /3/, and above we have discussed the high-frequency processes in FMF in which the internal moments must be taken into account. However, in many hydrodynamic problems the influence of the internal moments is not important; therefore the relaxation models of FMF in which the internal angular momenta are not taken into account, but the processes of relaxation of magnetization of the fluid are, are of interest. Below we consider the relaxation model of MF which is described, in the inertial Cartesian coordinate system, by the following set of equations:

$$\begin{gathered}\rho\,(d/dt)\,(v^{\alpha} - {}^{1}/{}_{2}\,\gamma\rho^{-1}\,\mathrm{rot}^{\alpha} M) = \partial_{\beta}\,[-p\delta^{\alpha\beta} + \tau^{\alpha\beta} + {}^{1}/{}_{2}\,(M^{\alpha}L^{\beta} - M^{\beta}L^{\alpha})] - \\ \gamma M^{\lambda}\,\partial^{\alpha}\omega_{\lambda} + {}^{1}/{}_{2}\,\gamma\,(\mathrm{rot}^{\lambda} M)\,\partial^{\alpha} v_{\lambda} + {}^{1}/{}_{2}\,(M^{\lambda}\,\partial^{\alpha} H_{\lambda} - H_{\lambda}\,\partial^{\alpha} M^{\lambda}) + Q^{\alpha}\end{gathered} \tag{5.1}$$

$$\mathrm{rot}\, H = 0, \ \mathrm{div}\,(H + 4\pi M) = 0$$

$$H^{\alpha} - \frac{1}{\chi} M^{\alpha} - \gamma\omega^{\alpha} = L^{\alpha}, \quad \frac{1}{\chi} = \frac{\rho}{M}\frac{\partial U}{\partial M}, \quad \frac{d\rho}{dt} + \rho\,\mathrm{div}\, v = 0$$

$$p = \rho^{2}\frac{\partial U}{\partial \rho} + \rho M\,\frac{\partial U}{\partial M} - \frac{1}{2} M^{\alpha} H_{\alpha}, \quad T = \frac{\partial U}{\partial s}$$

$$T\,\frac{ds}{dt} = -\,\partial_{\alpha} q^{\alpha} + \tau^{\alpha\beta} e_{\alpha\beta} + L^{\alpha}\left(\rho\,\frac{d}{dt}\,\frac{1}{\rho}\,M_{\alpha} - [\omega, M]_{\alpha}\right)$$

Here U is a given differentiable function of the parameters ρ, s and $M = (M_\alpha M^\alpha)^{1/2}$.
System (5.1) is obtained from the variational equation /15/

$$\delta \iint \Lambda \, dV_3 \, dt + \delta W^* + \delta W = 0$$

with the Lagrangian Λ and functional δW^* of the form

$$\Lambda = \frac{1}{2}\rho v^2 - \frac{1}{8\pi} H^2 - \gamma M^\alpha \omega_\alpha - \rho U(\rho, s, M)$$
$$\delta W^* = \iint \left(\rho T \delta s - \tau^{\alpha\beta} \partial_\beta \delta x_\alpha - \rho L^{\wedge\alpha} \delta \frac{1}{\rho} M_\alpha{}^\wedge\right) dV_3 \, dt$$

and represents a special case of Eqs.(1.1), (1.2).

System (5.1) is obtained using the variational equation from /3, 5/. Later, the equations were derived in /10/ using the energy equation, with the termodynamic fluxes $q^\alpha, \tau^{\alpha\beta}, L^\alpha$ depending linearly on the thermodynamic forces $\partial_\alpha T, e_{\alpha\beta}, dM/dt - [\omega, M]$ for $\gamma = 0$.

The relaxation term L^α of the equation for the magnetization, the components of the heat flux vector q^α and the components of the viscous stress tensor $\tau^{\alpha\beta}$ in Eqs.(5.1) can be determined using the dissipation function σ depending on the thermodynamic forces $e_{\alpha\beta}, \partial_\alpha T$, $\rho d(M/\rho)/dt - [\omega, M]$, and possibly on other defining parameters of the fluid and field

$$T\sigma = -\frac{1}{T} q^\alpha \partial_\alpha T + \tau^{\alpha\beta} e_{\alpha\beta} + L^\alpha \left(\rho \frac{d}{dt} \frac{1}{\rho} M_\alpha - [\omega, M]_\alpha\right) \tag{5.2}$$
$$L^\alpha = \mu_1 \partial\sigma / \partial \left(\rho \frac{d}{dt} \frac{1}{\rho} M_\alpha - [\omega, M]_\alpha\right)$$
$$\tau^{\alpha\beta} = \mu_2 \partial\sigma/\partial e_{\alpha\beta}, \quad q^\alpha = \mu_3 \, \partial\sigma/\partial\partial_\alpha T$$

where μ_1, μ_2, μ_3 are scalar functions of the defining parameters.

If σ is a quadratic function of thermodynamic forces, then Eqs.(5.2) will give the Onsager relations. In particular, in the simplest case relations (1.3), (1.4) can be used as the Onsager relations. Further we shall use for $\tau^{\alpha\beta}, L^\alpha$ a relation of the form

$$\tau^{\alpha\beta} = 2\mu e^{\alpha\beta} + \lambda \delta^{\alpha\beta} e_\lambda{}^\lambda, \quad L^\alpha = \theta^{\alpha\beta} \left(\rho \frac{d}{dt} \frac{1}{\rho} M_\beta - [\omega, M]_\beta\right) \tag{5.3}$$
$$\theta^{\alpha\beta} = \theta(\delta^{\alpha\beta} - n^\alpha n^\beta) + \theta_\parallel n^\alpha n^\beta$$

in which λ, μ are the coefficients of viscosity. The coefficients $\lambda, \mu, \theta, \theta_\parallel$ can, in general, be specified as functions of the defining parameters of the fluid and field.

In the case of an incompressible fluid the pressure p is regarded as an additional unknown function, and the equation for p in (5.1) is, in this case, neglected. Eqs.(5.1) take into account the gyromagnetic energy of the fluid, the magnetization relaxation processes, viscosity and thermal conductivity of the fluid.

The solutions of the Couette and Poiseuille problems obtained below show, that the system of Eqs.(5.1), (5.3) describes the effect of an increase in the viscosity of the fluid in a magnetic field, and when the coefficients θ in (5.3) are suitably chosen, the increase in the coefficient of viscosity is found to be the same as in the theory of FMF with internal rotation /8/.

6. Couette flow. Let us consider a stationary flow of fluid between the planes $x^3 = 0$, $x^3 = d = \text{const}$ in a given constant external field with induction $B^\circ = (B_1^\circ, B_2^\circ, B_3^\circ)$. The plane $x^3 = 0$ is fixed, and the plane $x^3 = d$ moves with constant velocity $v = (2\omega_0 d, 0, 0)$, $\omega_0 = \text{const}$. When $Q^\alpha = 0$, Eqs.(5.1) and (5.3) and the boundary conditions (the condition of adhesion of the fluid and the known condition for the magnetic field) are satisfied, if we put $M^\alpha = \text{const}$ in the region $0 < x^3 < d$ and

$$v = (2\omega_0 x^3, 0, 0), \; H = (B_1^\circ, B_2^\circ, B_3^\circ - 4\pi M_3)$$

The equation for magnetization in (5.1) is transformed, in the present case, to an algebraic equation, from which we obtain, up to terms of the first order of smallness in $\omega_0\theta$, assuming that the dimensionless parameter $\omega_0\theta$ is small,

$$M^\alpha = \chi H^\alpha + \theta \chi^2 [\omega, H]^\alpha - \chi\gamma\omega^\alpha$$

Determining the components of the force vector f_1 acting on the plane $x^3 = 0$, we find that $f^1 = 2\mu_e\omega_0$, where we have the following expression for the effective coefficient of viscosity μ_e:

$$\mu_e = \mu + {}^1/_4 \chi^2 H^2 \theta \sin^2 \alpha \tag{6.1}$$

Here α is the angle between the vorticity vector and the magnetic field vector H.

7. Poiseuille flow. Let us consider a stationary flow of fluid in a cylindrical pipe of circular cross-section of radius R, made of a non-magnetic material, under the action of a pressure drop, in a constant magnetic field H_0 directed along the pipe axis. We introduce a cylindrical system of coordinates r, φ, z, attached to the pipe, and we shall seek a solution of Eqs.(5.1) and (5.3) for $Q^\alpha = 0$ in the form of expansions in terms of the dimensionless parameter $\varepsilon = -{}^1/_4\rho R^3\mu^{-2}\partial p/\partial z$, assuming that ε is small

$$v = v_1\varepsilon + v_2\varepsilon^2 + \ldots,\ M = M_0 + M_1\varepsilon + M_2\varepsilon^2 + \ldots \tag{7.1}$$

Taking into account terms of the second order of smallness in ε, we obtain from (5.1), (5.3) the following equations for determining the functions v, M:

$$\begin{aligned}
&\mu\frac{d}{dr}v_1^z - \frac{1}{2}H_0(1+4\pi\chi)M_1^r = -\frac{2\mu^2}{\rho R^3}r \\
&\mu\left(\frac{d}{dr}v_2^\varphi - \frac{1}{r}v_2^\varphi\right) = \left(2\pi M_1^\varphi + \frac{1}{4}\gamma\frac{d}{dr}v_1^z\right)M_1^r, \\
&M_1^\varphi = \frac{1}{2}\chi\gamma\frac{d}{dr}v_1^z \\
&M_2^z\left(1 - H_0\frac{\partial\chi}{\partial M_0}\right) = -\frac{\chi\gamma}{2}\left(\frac{\partial v_2^\varphi}{\partial r} + \frac{v_2^\varphi}{r}\right) + \frac{\chi\theta}{2}M_1^r\frac{dv_1^z}{dr} + \\
&\quad\frac{1}{2\chi}\frac{\partial\chi}{\partial M_0}[(M_1^r)^2 + (M_1^\varphi)^2] \\
&\quad\frac{1+4\pi\chi}{\chi}M_1^r + \frac{\theta}{2}M_0^z\frac{d}{dr}v_1^z = 0
\end{aligned} \tag{7.2}$$

Eqs.(7.1), (7.2) have the following solution satisfying the condition of adhesion of the fluid at the pipe walls:

$$\begin{aligned}
&v^r = 0,\ v^z = \frac{1}{4\mu_e}(r^2 - R^2)\frac{\partial p}{\partial z}, \quad v^\varphi = \frac{\gamma(\mu-\mu_e)}{16\mu\mu_e^2 H_0}\left(\frac{\partial p}{\partial z}\right)^2 r(r^2 - R^2) \\
&M^\varphi = \frac{\chi\gamma}{4\mu_e}r\frac{\partial p}{\partial z}, \quad M^r = -\frac{1}{H_0(1+4\pi\chi)}\frac{\mu_e - \mu}{\mu_e}r\frac{\partial p}{\partial z} \\
&M^z = \chi H_0 + \frac{(\partial p/\partial z)^2}{1 - H_0\partial\chi/\partial M_0}\left\{\frac{1}{2\chi}\frac{\partial\chi}{\partial M_0}\left(\frac{\chi^2\gamma^2}{16\mu_e^2} + \frac{(\mu_e - \mu)^2}{H_0^2(1+4\pi\chi)^2\mu_e^2}\right)r^2 + \right. \\
&\quad\left.\frac{\chi(\mu-\mu_e)}{4H_0\mu_e^2}\left[\frac{\gamma^2}{4\mu}(-2r^2 + R^2) + \frac{\theta}{1+4\pi\chi}r^2\right]\right\}
\end{aligned}$$

Here $\mu_e = \mu + {}^1/_4\chi^2 H_0^2\theta$ is the effective coefficient of viscosity (identical with expression (6.1), since we have here $\alpha = \pi/2$ to within the degree of accuracy used).

The solution shows that when the gyromagnetic energy (at $\gamma \neq 0$) is taken into account, the magnetized fluid executes a screw motion within the pipe. The rate of fluid flow is described by the usual relation of the Navier-Stokes theory of a viscous fluid in which the coefficient of viscosity μ is replaced by the effective coefficient μ_e.

We note that an estimate of the terms with coefficient γ in the solutions obtained above shows that in the case of real fluids these terms are, in general, small, and their influence on the flows discussed here can be disregarded in practice. However, these solutions with small terms containing γ describe effects (e.g. the magnetization of the fluid determined by the vorticity) which can be utilized in the experimental determination of the quantity γ. On the other hand, it is clear that hydrodynamic problems exist (e.g. those connected with supersonic waves), in which the terms with γ are appreciable.

The form of the dependence of the magnetization relaxation time θ in formula (6.1) on the defining parameters, and in particular on the magnetic field, must be chosen by comparing it with the experimental relation $\mu_e = \mu_e(H)$. In particular, formula (6.1), which can be applied to the FMF already when $\theta = \text{const}$, describes the saturation of the viscosity in FMF in strong magnetic fields, whose magnitude corresponds exactly to the experimental value when the constant θ is chosen approximately (in the FMF the relaxation time θ is determined by the characteristic rotational Brownian diffusion of the particles in the FMF, and for particles of diameter ~ 100 Å in a fluid of viscosity $\sim 10^{-2}$ Poise it is of the order of $\sim 10^{-6}$ sec.). When the magnetic fields are weak, formula (6.1) at $\theta = \text{const}$ describes quantitatively correctly, in any case, existing experimental data /16-18/. A quantitative comparison with these experiments in the region of weak fields is difficult, since /16-18/ contain no direct experimental data showing the dependence of the magnetization of the FMF on the field. We also note that the formula in /8/ for μ_e can also be obtained from (6.1) by choosing, in an appropriate manner, the dependence of θ on the field. However, according to /17, 18/ the relation for μ_e shows a poor agreement with experiment (the particle concentrations used in /8/ to obtain the correct values of μ_e in the region of saturation, do not correspond to the actual concentrations).

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ON THE PAPER BY V.A. ZHELNOROVICH ENTITLED "ON MATHEMATICAL MODELS OF MAGNETIC FLUIDS"*

V.V. GOGOSOV

The paper by V.V. Zhelnorovich /1/ consists of two parts. In the first part equations are given for describing magnetic fluids (MF), which are not basically different from the well-known euqations /2-4/. The system is linearized and a solution is sought for the propagation of monochromatic waves. In the second part of the paper (Sect.5) an attempt is made to reply to a criticism which appeared in the review /4/ and in /3/, concerning the paper by Zhelnorovich /5/. Further, another model of MF is described in Sect.5, different from that given in Sect.1 of /1/, and is used to solve the Couette and Poiseuille problems. In doing this he not only repeats the errors already discussed in /3, 4/, but he also makes further errors, which will be discussed in the present paper.

The interest in describing the behaviour of magnetizable fluid media in nagnetic fields is primarily connected with producing MF and practical applications of MF, which are colloidal solutions of fine ferromagnetic particles.

It would seem that the simplest way of describing the behaviour of MF would be to use the normal equations of hydrodynamics with an additional force $\mathbf{M}\nabla\mathbf{H}$ acting on the fluid from the direction of the magnetic field $\mathbf{H}$. The magnetization $\mathbf{M}$ can, in most cases, be assumed to be parallel to the magnetic field $\mathbf{M} = \chi\mathbf{H}$. Precisely such a model was proposed in /6/. It satisfactorily describes many phenomena and is widely used in practice.

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